ME-221
SOLUTIONS FOR PROBLEM SET 4

Problem 1

Let's define:

$$\begin{aligned}
\dot{x_1} &= x_1 + x_2 - (u_1 + u_2) = f_1(x,u) \\
\dot{x_2} &= x_1^2 - (x_2 - 1)^2 + x_1 x_2 - u_1^2 - u_2 = f_2(x,u) \\
y_1 &= x_1(1 + x_2) + u_1 = g_1(x,u) \\
y_2 &= x_1 + x_2 - u_2 = g_2(x,u)
\end{aligned}$$

At the equilibrium point corresponding to $\bar{u}_1 = \bar{u}_2 = 1$, the variables $\bar{x}_1$ and $\bar{x}_2$ satisfy the following relations:

$$\begin{aligned}
0 &= \bar{x}_1 + \bar{x}_2 - 2 \\
0 &= \bar{x}_1^2 - (\bar{x}_2 - 1)^2 + \bar{x}_1 \bar{x}_2 - 2
\end{aligned}$$

Solving these equations while taking the condition of $\bar{x}_1 > 0$ and $\bar{x}_2 > 0$ (given in the question) into account, we find that $\bar{x}_1 = \bar{x}_2 = 1$.

We can linearize the non-linear system around the equilibrium point using the Jacobian approach. After taking the partial derivatives of $f_1(x,u)$, $f_2(x,u)$, $g_1(x,u)$, and $g_2(x,u)$ at the equilibrium point, we obtain the A, B, C and D matrices for the state-space representation.

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{u},\bar{x}} = \begin{bmatrix} 1 & 1 \\ 2\bar{x}_1 + \bar{x}_2 & -2(\bar{x}_2 - 1) + \bar{x}_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{\bar{u},\bar{x}} = \begin{bmatrix} -1 & -1 \\ -2\bar{u}_1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$C = \left.\frac{\partial g}{\partial x}\right|_{\bar{u},\bar{x}} = \begin{bmatrix} 1 + \bar{x}_2 & \bar{x}_1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$D = \left.\frac{\partial g}{\partial u}\right|_{\bar{u},\bar{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

With these matrices the state model can be constructed as:

$$\begin{aligned}
\delta\dot{x} &= A\delta x + B\delta u \\
\delta y &= C\delta x + D\delta u
\end{aligned}$$

where $\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$ and $\delta y = y - \bar{y}$.

We obtain the state-space representation of the linearized system:

$$
\begin{aligned}
\delta\dot{x_1} &= \delta x_1 + \delta x_2 - \delta u_1 - \delta u_2 \\
\delta\dot{x_2} &= 3\delta x_1 + \delta x_2 - 2\delta u_1 - \delta u_2 \\
\delta y_1 &= 2\delta x_1 + \delta x_2 + \delta u_1 \\
\delta y_2 &= \delta x_1 + \delta x_2 - \delta u_2
\end{aligned}
$$

Problem 2

a)

Newton's method By applying Newton's 2nd law, we obtain the equation of motion:

$$ml^2\ddot{\theta}(t) = -lsin(\theta(t))mg - acos(\theta(t))F_s + lF(t) \quad (1)$$

With $F_s = k\Delta x$ being the force applied by the spring, where k is the spring constant and Δx is the elongation of the spring. Given that $\theta = 0$ corresponds to the position at which the spring is in its relaxed state, $\Delta x = asin\theta$.

We can therefore rewrite 1 as:

$$\ddot{\theta}(t) = -\frac{g}{l}sin(\theta(t)) - \frac{a^2k}{ml^2}sin(\theta(t))cos(\theta(t)) + \frac{1}{ml}F(t) \quad (2)$$

The state and input variables can be chosen as follows:

$x_1 = \theta$, $x_2 = \dot{\theta}$, $u = F$

This selection leads to the following state-space model:

$$\dot{x_1} = x_2 \qquad x_1(0) = \theta_0$$
$$\dot{x_2} = -\frac{g}{l}sinx_1 - \frac{a^2k}{ml^2}sinx_1cosx_1 + \frac{1}{ml}u \qquad x_2(0) = \omega_0$$
$$y = x_1$$

Lagrange's method The same result can be obtained by applying Lagrange's method.
The kinetic energy of the pendulum is:

$$T = \frac{1}{2}m(l\dot{\theta})^2 \quad (3)$$

The elongation of the spring is $asin(\theta)$. Therefore the total potential energy can be written as:

$$V = \frac{1}{2}k(asin(\theta))^2 - mglcos(\theta) \quad (4)$$

The Lagrangian is:

$$L = T - V = \frac{1}{2}m(l\dot{\theta})^2 - \frac{1}{2}k(asin(\theta))^2 + mglcos(\theta) \quad (5)$$

We then calculate:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = M \tag{6}$$

where $M = lF$ is the external moment applied to the system.

We get:

$$ml^2\ddot{\theta} + ka^2 sin(\theta)cos(\theta) + mglsin(\theta) = lF \tag{7}$$

$$\ddot{\theta} = \frac{F}{ml} - \frac{a^2k}{ml^2}sin(\theta)cos(\theta) - \frac{g}{l}sin(\theta) \tag{8}$$

b) The nonlinear model can be linearized for small rotations around the vertical position ($\bar{x_1} = 0$). We then obtain the following linearized state model:

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}cos\bar{x_1} - \frac{a^2k}{ml^2}(-sin^2\bar{x_1} + cos^2\bar{x_1}) & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{ml} \end{bmatrix} \delta u = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} - \frac{a^2k}{ml^2} & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{ml} \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

With $\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$ and $\delta y = y - \bar{y}$

Problem 3

a) The state variables, input and output can be chosen as follows:

$x_1 = x$, $x_2 = \dot{x}$, $u = i$ and $y = x$

We then obtain the following state-space model:

$$\dot{x_1} = x_2 = f_1(x, u)$$
$$\dot{x_2} = g - \frac{L}{2m(1+x_1)^2}u^2 = f_2(x, u)$$
$$y = x_1 = g_1(x, u)$$

b) We can linearize the non-linear system around the equilibrium point using the Jacobian approach. After taking the partial derivatives of $f_1(x, u)$, $f_2(x, u)$, and $g_1(x, u)$ at the equilibrium point, we obtain the A, B, C and D matrices for the state-space representation.

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ \dfrac{L\bar{u}^2}{m(1+\bar{x}_1)^3} & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ -\dfrac{L\bar{u}}{m(1+\bar{x}_1)^2} \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

With $\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$ and $\delta y = y - \bar{y}$

Problem 4

Applying Newton's second law we obtain:

$$
\begin{aligned}
m\ddot{y} &= -b(\dot{y} - \dot{u}) - k(y - u) \\
m\ddot{y} + b\dot{y} + ky &= b\dot{u} + ku \\
\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y &= \frac{b}{m}\dot{u} + \frac{k}{m}u
\end{aligned}
$$

To take care of the input derivative term, we will use the variable change trick:

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

With:

$$a_1 = \frac{b}{m}, \quad a_2 = \frac{k}{m}, \quad b_0 = 0, \quad b_1 = \frac{b}{m}, \quad b_2 = \frac{k}{m}$$

And define:

$$
\begin{aligned}
\beta_0 &= b_0 = 0 \\
\beta_1 &= b_1 - a_1\beta_0 = b_1 = \frac{b}{m} \\
\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - (\frac{b}{m})^2
\end{aligned}
$$

Now we have everything to obtain the state-space representation of the dynamical system.

$$
\begin{aligned}
x_1 =& y - \beta_0 u = y \\
x_2 =& \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u \\
\dot{x}_1 =& x_2 + \beta_1 u = x_2 + \frac{b}{m}u \\
\dot{x}_2 =& -a_2 x_1 - a_1 x_2 + \beta_2 u \\
=& -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - (\frac{b}{m})^2\right]u
\end{aligned}
$$